

A STRONG SZEGŐ THEOREM FOR JACOBI MATRICES

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ABSTRACT. We use a classical result of Gollinski and Ibragimov to prove an analog of the strong Szegő theorem for Jacobi matrices on $l^2(\mathbb{N})$. In particular, we consider the class of Jacobi matrices with conditionally summable parameter sequences and find necessary and sufficient conditions on the spectral measure such that $\sum_{k=n}^{\infty} b_k$ and $\sum_{k=n}^{\infty} (a_k^2 - 1)$ lie in l_1^2 , the linearly-weighted l^2 space.

1. INTRODUCTION

Let us begin with some notation. We study the spectral theory of Jacobi matrices, that is semi-infinite tridiagonal matrices

$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 \\ a_1 & b_2 & a_2 & 0 \\ 0 & a_2 & b_3 & \ddots \\ 0 & 0 & \ddots & \ddots \end{pmatrix}$$

where $a_n > 0$ and $b_n \in \mathbb{R}$. In this paper we make the overarching assumption that the sequences b_n and $a_n^2 - 1$ are conditionally summable. We may then define

$$(1.1) \quad \begin{aligned} \lambda_n &:= - \sum_{k=n+1}^{\infty} b_k \\ \kappa_n &:= - \sum_{k=n+1}^{\infty} (a_k^2 - 1) \end{aligned}$$

for $n = 0, 1, \dots$.

Let $d\nu$ be the spectral measure for the pair (J, δ_1) , where $\delta_1 = (1, 0, 0, \dots)^t$, and assume that $d\nu$ is not supported on a finite set of points (we will call such measures *nontrivial*). Let

$$(1.2) \quad m(z) := \langle \delta_1, (J - z)^{-1} \delta_1 \rangle = \int \frac{d\nu(x)}{x - z}$$

be the associated m -function, defined for $z \in \mathbb{C} \setminus \text{supp}(\nu)$.

We will write

$$\{\beta_n\} \in l_s^2 \quad \text{if} \quad \|\beta\|_{l_s^2}^2 := \sum_n |n|^s |\beta_n|^2 < \infty,$$

and let $\dot{H}^{1/2}(\partial\mathbb{D})$ denote the (homogeneous) Sobolev space of order $1/2$ of functions defined on $\partial\mathbb{D}$:

$$f \in \dot{H}^{1/2} \quad \text{if} \quad \|f\|_{\dot{H}^{1/2}}^2 := \|\hat{f}(n)\|_{l_1^2}^2 = \sum_n |n| |\hat{f}(n)|^2 < \infty.$$

If f is a function on $[-2, 2]$, we say $f \in \dot{H}^{1/2}$ if $f(2 \cos \theta) \in \dot{H}^{1/2}(\partial \mathbb{D})$. Also, we will say $v \in \mathcal{W}$ if $v(x)$ is supported in $[-2, 2]$ and has one of the forms

$$(1.3) \quad \left(\sqrt{4 - x^2} \right)^{\pm 1} v_0(x) \quad \text{or} \quad \left(\sqrt{\frac{2 - x}{2 + x}} \right)^{\pm 1} v_0(x)$$

with $\log(v_0) \in \dot{H}^{1/2}$.

Our main result is:

Theorem 1.1. *Let J be a Jacobi matrix. The following are equivalent:*

- (1) *The sequences associated to J by (1.1) obey $\lambda, \kappa \in l_1^2$*
- (2) *J has finitely-many eigenvalues that all lie in $\mathbb{R} \setminus [-2, 2]$, and on $[-2, 2]$ the spectral measure is purely absolutely continuous, $d\nu(x) = v(x)dx$, with $v \in \mathcal{W}$.*

The main ingredient in the proof will be the following version of the strong Szegő theorem¹.

Theorem 1.2 (Golinskii-Ibragimov). *Let $d\mu$ be a probability measure on $\partial \mathbb{D}$ that is not supported on a finite set of points, and let $\{\alpha_n\} \subseteq \mathbb{D}$ be the associated Verblunsky coefficients. The following are equivalent:*

- (1) $\alpha \in l_1^2$
- (2) $d\mu = w \frac{d\theta}{2\pi}$ and $\log w \in \dot{H}^{1/2}$.

We now outline the proof of Theorem 1.1. To apply the strong Szegő theorem we must move to the circle, so we must first remove all the eigenvalues in $\mathbb{R} \setminus [-2, 2]$. To do so we use double commutation (see [5]):

Theorem 1.3 (Double Commutation). *Let $E \in \mathbb{R} \setminus \sigma(J)$, and let $\gamma > 0$. Define a new Jacobi matrix \tilde{J} by*

$$\begin{aligned} \tilde{a}_n &= a_n \frac{\sqrt{c_{n-1}c_{n+1}}}{c_n} \\ \tilde{b}_n &= b_n + \gamma \left(\frac{a_{n-1}\phi_{n-1}\phi_n}{c_{n-1}} - \frac{a_n\phi_n\phi_{n+1}}{c_n} \right) \end{aligned}$$

where $J\phi = E\phi$, $\phi_0 = 0$, $\phi_1 = 1$ and

$$c_n = 1 + \gamma \sum_{j=1}^n |\phi_j|^2.$$

Then $\sigma(\tilde{J}) = \sigma(J) \cup \{E\}$, E is a simple eigenvalue of \tilde{J} , and

$$\tilde{m}(z) = \frac{1}{1 + \gamma} \left(m(z) - \frac{\gamma}{z - E} \right).$$

Conversely, let $|E| > 2$ be a simple eigenvalue of J with eigenvector ϕ . Choose $\gamma = -1/\|\phi\|^2$ and define a new Jacobi matrix \tilde{J} as above. Then $\sigma(\tilde{J}) = \sigma(J) \setminus \{E\}$ and

$$\tilde{m}(z) = (1 + \gamma)m(z) + \frac{\gamma}{z - E}.$$

We prove an asymptotic integration result in Section 2, which we combine with the above theorem in Section 3 to prove

¹The version we use is due to [7] and [11]. For relevant definitions see, for instance, [16].

Proposition 1.4. *Let J be a Jacobi matrix, and let E be an isolated eigenvalue of J in $\mathbb{R} \setminus [-2, 2]$. Let \tilde{J} be the Jacobi matrix obtained from J by removing the eigenvalue E using Theorem 1.3. Then*

- (1) $\tilde{\lambda}, \tilde{\kappa} \in l_1^2$ if and only if $\lambda, \kappa \in l_1^2$
- (2) $\tilde{v} \in \mathcal{W}$ if and only if $v \in \mathcal{W}$.

This proposition essentially allows us to consider Theorem 1.1 under the additional hypothesis $\sigma(J) \subseteq [-2, 2]$. This allows us to move to the circle, as follows. Given a nontrivial probability measure $d\mu$ on $\partial\mathbb{D}$ that is invariant under complex conjugation, one can define a nontrivial probability measure $d\nu$ on $[-2, 2]$ by

$$\int_{-2}^2 g(x) d\nu(x) = \int_0^{2\pi} g(2 \cos \theta) d\mu(\theta).$$

Similarly, given such a measure $d\nu$, one can define a measure $d\mu$ that is symmetric under complex conjugation by

$$\int_0^{2\pi} h(\theta) d\mu(\theta) = \int_{-2}^2 h(\arccos(x/2)) d\nu(x)$$

when $h(-\theta) = h(\theta)$.

The map $d\mu \mapsto d\nu$ is one of a family of four maps that we call the Szegő mappings². We denote it by $d\nu = Sz^{(e)}(d\mu)$. The other three maps are given by

$$(1.4) \quad \begin{aligned} Sz^{(o)}(d\mu) &= c^2(4 - x^2)Sz^{(e)}(d\mu) \\ Sz^{(\pm)}(d\mu) &= c_{\pm}^2(2 \mp x)Sz^{(e)}(d\mu) \end{aligned}$$

$$(1.5) \quad \begin{aligned} c &= \frac{1}{\sqrt{2(1 - |\alpha_0|^2)(1 - \alpha_1)}} \\ c_{\pm} &= \frac{1}{\sqrt{2(1 \mp \alpha_0)}}. \end{aligned}$$

If $d\mu$ is absolutely continuous with respect to Lebesgue measure we will write $d\mu(\theta) = w(\theta) \frac{d\theta}{2\pi}$ and then $Sz^{(*)}(d\mu)(x) = v^{(*)}(x)dx$. In this case the above relations become

$$(1.6) \quad \begin{aligned} v^{(e)}(x) &= \frac{1}{\pi\sqrt{4 - x^2}}w(\arccos(x/2)) \\ v^{(o)}(x) &= \frac{c}{\pi}\sqrt{4 - x^2}w(\arccos(x/2)) \\ v^{(\pm)}(x) &= c_{\pm}\sqrt{\frac{2 \mp x}{2 \pm x}}w(\arccos(x/2)). \end{aligned}$$

For $*$ in $\{e, o, +, -\}$, we will write $J^{(*)}$ for the Jacobi matrix determined by $d\nu^{(*)}$ and $a^{(*)}, b^{(*)}$ for its parameter sequences. The relationship between α and $a^{(*)}, b^{(*)}$ is given by

Proposition 1.5 (Direct Geronimus Relations³). *Let $d\mu$ be a nontrivial probability measure on $\partial\mathbb{D}$ that is invariant under conjugation, and let $d\nu^{(*)} = Sz^{(*)}(d\mu)$. Then*

²The map $Sz^{(e)}$ is due to [17], while the other three are due to [2], then developed further in [13] and [16].

³The relationship between α and $a^{(e)}, b^{(e)}$ was first discovered by [4]. The other three were later found by [2] using techniques similar to [4]. [13] and [16] have a different proof using operator techniques.

for all $n \geq 0$

$$\begin{aligned}
[a_{n+1}^{(e)}]^2 &= (1 - \alpha_{2n-1})(1 - \alpha_{2n}^2)(1 + \alpha_{2n+1}) \\
b_{n+1}^{(e)} &= \alpha_{2n}(1 - \alpha_{2n-1}) - \alpha_{2n-2}(1 + \alpha_{2n-1}) \\
[a_{n+1}^{(o)}]^2 &= (1 + \alpha_{2n+1})(1 - \alpha_{2n+2}^2)(1 - \alpha_{2n+3}) \\
b_{n+1}^{(o)} &= -\alpha_{2n+2}(1 + \alpha_{2n+1}) + \alpha_{2n}(1 - \alpha_{2n+1}) \\
[a_{n+1}^{(\pm)}]^2 &= (1 \pm \alpha_{2n})(1 - \alpha_{2n+1}^2)(1 \mp \alpha_{2n+2}) \\
b_{n+1}^{(\pm)} &= \mp \alpha_{2n+1}(1 \pm \alpha_{2n}) \pm \alpha_{2n-1}(1 \mp \alpha_{2n}).
\end{aligned}$$

Since $a_n > 0$, there is no ambiguity in which sign to choose for the square root above. We always take $\alpha_{-1} = -1$. The value of α_{-2} is irrelevant since it is multiplied by zero.

From the Direct Geronimus Relations we see that decay of the α 's determines decay of the a 's and b 's. This allows us to prove one direction of Theorem 1.1 in Section 4.

To prove the other direction, we will find certain relationships between the Verblunsky parameters and solutions of $Ju = Eu$ at $E = \pm 2$. We study asymptotics of these solutions in Section 5, then find the desired relationships in Section 6, which we term the Inverse Geronimus Relations. In Section 7 we review some Weyl theory, and in Section 8 we combine all these ideas to finish the proof.

It is a pleasure to thank Rowan Killip for his helpful advice.

2. ASYMPTOTIC INTEGRATION

Suppose \tilde{J} and J are related through double commutation (as in Theorem 1.3). In the next section we will relate $\tilde{\lambda}, \tilde{\kappa}$ to λ, κ . By Theorem 1.3 we see⁴

$$\begin{aligned}
|\tilde{\kappa}(n-1) - \kappa(n-1)| &= \left| \sum_{k=n}^{\infty} a(k)^2 \left(\frac{c(k-1)c(k+1)}{c(k)^2} - 1 \right) \right| \\
|\tilde{\lambda}(n-1) - \lambda(n-1)| &= |\gamma| \left| \sum_{k=n}^{\infty} \left(\frac{a(k-1)\phi(k-1)\phi(k)}{c(k-1)} - \frac{a(k)\psi(k)\psi(k+1)}{c(k)} \right) \right|.
\end{aligned}$$

So to prove part (1) of Proposition 1.4, we must determine asymptotics for ϕ when $E \in \mathbb{R} \setminus [-2, 2]$. To do so we use the theory of asymptotic integration as developed in [8, 9, 10, 14] and particularly [1]. However, as we need l_s^p control of the errors (rather than the usual $o(1)$ control) we must modify their results. Throughout, we will use the notation $x \lesssim y$ if there is a constant $c > 0$ such that $x \leq cy$. Also, if x_n is a sequence, we write $x = y + l_s^p$ to indicate $x_n = y_n + \varepsilon_n$ for some other sequence $\varepsilon \in l_s^p$.

Proposition 2.1. *Let $\Lambda(k) = \text{diag}[\lambda_1(k), \dots, \lambda_n(k)]$ and suppose that there exists $0 < \delta < 1$ so that for a fixed i either*

$$(2.1) \quad (I) \quad \left| \frac{\lambda_i(k)}{\lambda_j(k)} \right| \geq 1 + \delta \quad \text{or} \quad (II) \quad \left| \frac{\lambda_i(k)}{\lambda_j(k)} \right| \leq 1 - \delta$$

⁴In order to avoid excessive subscripting later in this section, we will write $a(n)$ for a_n , etc.

for each $j \neq i$, where $k \geq k_0$ for some k_0 . Suppose also that $\|V(k)\| \in l_s^2$ for some $s \geq 0$. Then the system

$$(2.2) \quad \Psi(k+1) = [\Lambda(k) + V(k)]\Psi(k)$$

has a solution of the form

$$(2.3) \quad \Psi_i(k) = \left(\prod_{l=k_0}^{k-1} \lambda_i(l) + V_{ii}(l) \right) (e_i + l_s^2)$$

where e_i is the i^{th} standard unit vector in \mathbb{R}^n .

As all norms on a finite dimensional space are equivalent, it does not matter which we mean when we write things like $\|V(k)\| \in l_s^p$ or $(e_i + l_s^p)$.

We will prove Proposition 2.1 by using a Harris-Lutz transformation followed by a Levinson-type result. We will state and use these results, then prove them at the end of this section.

Proposition 2.2. *With the assumptions of Proposition 2.1, there exists a sequence of matrices $Q(k)$ such that $Q(k)_{ii} = 0$, $\|Q(k)\| \in l_s^2$, and*

$$(2.4) \quad V(k) - \text{diag}V(k) + \Lambda(k)Q(k) - Q(k+1)\Lambda(k) = 0.$$

Proposition 2.3. *Say $\Lambda(k)$ satisfies the assumptions of Proposition 2.1, and suppose that $\|R(k)\| \in l_s^1$ for some $s \geq 0$. Then the system*

$$(2.5) \quad x(k+1) = [\Lambda(k) + R(k)]x(k)$$

has a solution of the form

$$(2.6) \quad x_i(k) = \left(\prod_{l=k_0}^{k-1} \lambda_i(l) \right) (e_i + l_s^2).$$

Assuming Propositions 2.2 and 2.3 we have

Proof of Proposition 2.1. Let $Q(k)$ be as guaranteed by Proposition 2.2, and define $x(k)$ by

$$\Psi(k) = [I + Q(k)]x(k)$$

(as $Q(k) \rightarrow 0$, $[I + Q(k)]$ is invertible for large k , so the above definition makes sense). Then Ψ is a solution of (2.2) if and only if x solves

$$x(k+1) = [\tilde{\Lambda}(k) + \tilde{V}(k)]x(k)$$

where

$$\tilde{\Lambda}(k) = \Lambda(k) + \text{diag}V(k)$$

$$\tilde{V}(k) = [I + Q(k)]^{-1}[V(k)Q(k) - Q(k+1)\text{diag}V(k)].$$

It is easy to see that $\tilde{\Lambda}$ still satisfies the dichotomy condition (2.1). Moreover, as $\|V(k)\|, \|Q(k)\| \in l_s^2$ we have that $\|\tilde{V}(k)\| \in l_s^1$. So we may apply Proposition 2.3 to the x -system to find a solution

$$x_i(k) = \left(\prod_{l=k_0}^{k-1} \tilde{\lambda}_i(l) \right) (e_i + \varepsilon(k))$$

for some $\varepsilon(k) \in l_s^2$. But then

$$\Psi_i(k) = [I + Q(k)]x_i(k) = \left(\prod_{l=k_0}^{k-1} \lambda_i(l) + V(l)_{ii} \right) (e_i + \varepsilon(k) + Q(k)e_i + Q(k)\varepsilon(k)).$$

By Proposition 2.2 we have that $\|\varepsilon(k) + Q(k)e_i + Q(k)\varepsilon(k)\| \in l_s^2$, as required. \square

Next we prove Propositions 2.2 and 2.3. In doing so we will make frequent use of the following two lemmas.

Lemma 2.4. *Let $s \geq 1$, $\beta, \gamma \in l_s^2$, and define a sequence $\eta_n := \sum_{k=n}^{\infty} \beta_k \gamma_k$. Then $\eta \in l_s^2$ and $\|\eta\|_{l_s^2} \leq \|\beta\|_{l_s^2} \|\gamma\|_{l_s^2}$. In particular, if $\tau \in l_s^1$ then $\sum_{k=n}^{\infty} \tau_k \in l_s^2$.*

Proof. Throughout the proof, all norms refer to l_s^2 . By Cauchy-Schwarz we have

$$\begin{aligned} \|\eta\|^2 &= \sum_{n=1}^{\infty} n^s \left| \sum_{k=n}^{\infty} \beta_k \gamma_k \right|^2 \leq \sum_{n=1}^{\infty} n^s \left(\sum_{k=n}^{\infty} |\beta_k \gamma_k| \right)^2 \\ &\leq \sum_{n=1}^{\infty} n^s \left(\sum_{k=n}^{\infty} |\beta_k|^2 \right) \left(\sum_{k=n}^{\infty} |\gamma_k|^2 \right) = \sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} n^s |\beta_k|^2 \right) \left(\sum_{k=n}^{\infty} |\gamma_k|^2 \right) \\ &\leq \|\beta\|^2 \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} |\gamma_k|^2 = \|\beta\|^2 \sum_{k=1}^{\infty} k |\gamma_k|^2 \leq \|\beta\|^2 \|\gamma\|^2. \end{aligned}$$

The last statement follows by applying the above argument to $\beta = \gamma = |\tau|^{1/2}$. \square

Lemma 2.5. *Suppose $\gamma(k) \in l_1^2$ and $\beta > 1$. Then $\sum_{l=1}^{k-1} \beta^{2(l-k)} \gamma(l) \in l_1^2$.*

Proof. We must show that

$$\gamma \mapsto \sum_{l=1}^{k-1} \beta^{2(l-k)} \gamma(l)$$

maps $l_1^2 \rightarrow l_1^2$. Equivalently we will show

$$\gamma \mapsto \sum_{l=1}^{k-1} \sqrt{\frac{k}{l}} \beta^{2(l-k)} \gamma(l)$$

maps $l^2 \rightarrow l^2$. This is an integral operator with kernel

$$h(l, k) = \chi_{\{1, \dots, k-1\}}(l) \sqrt{\frac{k}{l}} \beta^{2(l-k)}$$

so by Schur's Test this will be a bounded operator if we can show

$$\sup_k \sum_{l=1}^{\infty} h(l, k) \leq C \quad \text{and} \quad \sup_l \sum_{k=1}^{\infty} h(l, k) \leq C$$

for some $C \geq 0$. This is done by the following lemma. \square

Lemma 2.6. *For any $\alpha \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$\sup_l \sum_{k=1}^{\infty} \left(\frac{|k|+1}{|l|+1} \right)^{\alpha} e^{-\varepsilon|k-l|} < \infty.$$

The proof is standard and proceeds by splitting the sum at $k = l$ and bounding each piece separately. We omit the details.

Proof of Proposition 2.2. Define $Q(k)$ by $Q(k)_{ii} = 0$ and

$$Q(k)_{ij} = - \sum_{m=k}^{\infty} \frac{V(m)_{ij}}{\lambda_j(m)} \prod_{l=k}^m \frac{\lambda_j(l)}{\lambda_i(l)} \quad \text{if } (i, j) \in (I)$$

$$Q(k)_{ij} = \sum_{m=k_0}^{k-1} \frac{V(m)_{ij}}{\lambda_i(m)} \prod_{l=k}^m \frac{\lambda_i(l)}{\lambda_j(l)} \quad \text{if } (i, j) \in (II).$$

As $\|V(k)\| \in l_s^p \subseteq l^\infty$, $Q(k)_{ij}$ is dominated (in either case above) by a convergent geometric series, so the sums defining Q converge. By the above definition, (2.4) holds.

To show that $\|Q(k)\| \in l_s^p$ we argue as follows. For $(i, j) \in (I)$ we have that

$$\left| \frac{1}{\lambda_j(m)} \prod_{l=k}^m \frac{\lambda_i(l)}{\lambda_j(l)} \right| \in l_1^2.$$

Similarly, for $(i, j) \in (II)$ we have that

$$\left| \frac{1}{\lambda_i(m)} \prod_{l=m}^{k-1} \frac{\lambda_i(l)}{\lambda_j(l)} \right| \lesssim |\beta|^{m-k}$$

for some $|\beta| > 1$. So by Lemmas 2.4 and 2.5 we see that $Q \in l_s^p$. \square

Proof of Proposition 2.3. Define $w(k)$ by

$$x(k) = \left(\prod_{l=k_0}^{k-1} \lambda_i(l) \right) w(k).$$

Then x solves (2.5) if and only if w solves the system

$$w(k+1) = \frac{1}{\lambda_i(k)} [\Lambda(k) + R(k)] w(k).$$

We'll compare the w -system to the diagonal system

$$y(k+1) = \frac{1}{\lambda_i(k)} \Lambda(k) y(k).$$

The y -system has a fundamental matrix

$$Y(k) = \text{diag} \left[\prod_{l=k_0}^{k-1} \frac{\lambda_1(l)}{\lambda_i(l)}, \dots, 1, \dots, \prod_{l=k_0}^{k-1} \frac{\lambda_n(l)}{\lambda_i(l)} \right]$$

with a 1 in the i^{th} spot. Let $P_1 = \text{diag}[p_1, \dots, p_n]$ where

$$p_j = \begin{cases} 1, & (i, j) \in (I) \\ 0, & (i, j) \in (II) \end{cases}$$

and let $P_2 = I - P_1$. By the assumptions on $\Lambda(k)$ we see that for $k_0 \leq l \leq k-1$, $\|Y(k)P_1Y(l+1)^{-1}\| \leq C$ and for $k_0 \leq k \leq l$, $\|Y(k)P_2Y(l+1)^{-1}\| \leq C$ for some $C > 0$.

Now let $k_1 \geq k_0$ to be chosen later, and consider the operator

$$[Tz](k) = \sum_{l=k_1}^{k-1} Y(k)P_1Y(l+1)^{-1} \frac{1}{\lambda_i(l)} R(l)z(l) - \sum_{l=k}^{\infty} Y(k)P_2Y(l+1)^{-1} \frac{1}{\lambda_i(l)} R(l)z(l)$$

acting on $l^\infty(\mathbb{N}, \mathbb{C}^n)$. Choose k_1 so that

$$2C\delta^{-1} \sum_{l=k_1}^{\infty} \|R(l)\| < \varepsilon < 1$$

(which is possible because $\|R(k)\| \in l_s^1$). Then we see that

$$\|Tz\|_{l^\infty} \leq \left(2C\delta^{-1} \sum_{l=k_1}^{\infty} \|R(l)\| \right) \|z\|_{l^\infty} \leq \varepsilon \|z\|_{l^\infty}$$

for all $z \in l^\infty$. Thus, $T : l^\infty \rightarrow l^\infty$ is a contraction. In particular, given $y \in l^\infty$, there exists a unique $w \in l^\infty$ solving $w = y + Tw$.

Say $y \in l^\infty$ and $w = y + Tw$. By the definition of T , y is a solution of the y -system if and only if w is a solution to the w -system. In particular this holds for $y = e_i$. It remains to show $w = y + l_1^2$, for which we consider each of the sums defining Tw separately. As $\|R(k)\| \in l_s^1 \subseteq l_s^2$ and $\|Y(k)P_1Y(l+1)^{-1}\frac{1}{\lambda_i(l)}w(l)\| \lesssim 1$, Lemma 2.5 shows the first sum is in l_1^2 . Similarly, because $\|R(k)\| \in l_1^1$, Lemma 2.4 shows that the second sum is in l_1^2 . \square

Finally, we allow perturbed diagonalizable systems, rather than just the perturbed diagonal systems of Proposition 2.1.

Proposition 2.7. *Suppose $A(k)$ has eigenvalues $\lambda_i(k)$ satisfying (2.1) and $\sup_k |\lambda_j(k)| \leq C$ for all j . Let $A(k) = S(k)^{-1}\Lambda(k)S(k)$ where $\Lambda(k) = \text{diag}[\lambda_1(k), \dots, \lambda_n(k)]$, and suppose that $S(k) \rightarrow S(\infty)$ where $S(\infty)$ is invertible and $\|S(k+1) - S(k)\| \in l_s^2$ for some $s \geq 0$. Finally, suppose $V \in l_s^2$. Then the system*

$$(2.7) \quad \Psi(k+1) = [A(k) + V(k)]\Psi(k)$$

has a solution of the form

$$\Psi_i(k) = S(k)^{-1} \left(\prod_{l=k_0}^{k-1} \lambda_i(l) + \tilde{V}(l)_{ii} \right) (e_i + l_s^2)$$

where

$$\tilde{V}(k) = S(k)V(k)S(k)^{-1} + (S(k+1) - S(k))(A(k) + V(k))S(k)^{-1}$$

so in particular $\|\tilde{V}(k)\| \in l_s^2$.

Proof. We'll reduce to the case of Proposition 2.1. Define $z(k) = S(k)\Psi(k)$, so Ψ is a solution of (2.7) if and only if z solves the system

$$(2.8) \quad z(k+1) = [\Lambda(k) + \tilde{V}(k)]z(k)$$

where \tilde{V} is as in the statement of the proposition. Now

$$\begin{aligned} \|\tilde{V}(k)\| &\lesssim \|V(k)\| + \|S(k+1) - S(k)\|(\|A(k)\| + \|V(k)\|) \\ &\lesssim \|V(k)\| + \|S(k+1) - S(k)\| \in l_s^2 \end{aligned}$$

because $S(\infty)$ is invertible and $\sup_{j,k} |\lambda_j(k)| \leq C$. So by Proposition 2.1, there exists a solution to (2.8) of the form

$$z_i(k) = \left(\prod_{l=k_0}^{k-1} \lambda_i(l) + \tilde{V}(l)_{ii} \right) (e_i + l_s^2).$$

Undoing the transformation we find a solution to (2.7) of the form

$$\Psi_i(k) = S(k)^{-1} z_i(k)$$

as desired. \square

3. THE DOUBLE COMMUTATION RESULT

In this section we prove Proposition 1.4.

Proof of Proposition 1.4(2). Suppose that \tilde{J} and J are related by double commutation at $E \in \mathbb{R}$ an isolated point of $\sigma(J)$. Write $d\nu(x) = v(x)dx$ and recall that Lebesgue almost everywhere

$$v(x) = \frac{1}{\pi} \operatorname{Im} m(x + i0).$$

By Theorem 1.3

$$\tilde{m}(z) = \frac{1}{1 + \gamma} \left(m(z) - \frac{\gamma}{z - E} \right).$$

But then

$$\tilde{v}(x) = \frac{1}{\pi} \operatorname{Im} \tilde{m}(x + i0) = \frac{1}{\pi(1 + \gamma)} \operatorname{Im} m(x + i0) = \frac{1}{1 + \gamma} v(x)$$

almost everywhere. Clearly $\tilde{v} \in \mathcal{W}$ if and only if $v \in \mathcal{W}$. \square

Part (1) is more difficult, and will take the rest of this section to prove. We will use the asymptotic integration results obtained in Section 2.

Lemma 3.1. *Write $E = \beta + \beta^{-1}$ with $|\beta| > 1$. The recurrence equation at E has solutions of the form*

$$\psi_{\pm}(k) = c_{\pm} \beta^{\pm k} (1 + l_1^2)$$

for some constants $c_{\pm} \in \mathbb{R} \setminus \{0\}$.

Proof. We will prove the result for $E > 2$, the proof for $E < -2$ being similar. We can write the recurrence equation

$$a(k+1)\psi(k+1) + (b(k) - E)\psi(k) + a(k)\psi(k-1) = 0$$

as the system

$$(3.1) \quad \Psi(k+1) = [A(k) + V(k)]\Psi(k)$$

where

$$\Psi(k) = \begin{bmatrix} \psi(k) \\ \psi(k-1) \end{bmatrix} \quad A(k) = \begin{bmatrix} \frac{E}{a(k+1)} - 1 & -1 \\ 1 & 0 \end{bmatrix} \quad V(k) = \begin{bmatrix} \frac{-b(k)}{a(k+1)} & 1 - \frac{a(k)}{a(k+1)} \\ 0 & 0 \end{bmatrix}.$$

Let

$$\lambda_{\pm}(k) = \frac{E \pm \sqrt{E^2 - 4a(k+1)^2}}{2a(k+1)} \quad \Lambda(k) = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}$$

$$S(k) = \frac{1}{\lambda_+ - \lambda_-} \begin{bmatrix} 1 & -\lambda_- \\ -1 & \lambda_+ \end{bmatrix}.$$

Then $A(k) = S(k)^{-1}\Lambda(k)S(k)$. If $|E| > 2$ and k is large enough, then $\Lambda(k)$ satisfies the dichotomy condition (2.1). It is easy to see that the rest of the hypotheses in Proposition 2.7 are satisfied for $s = 1$, so there are solutions of the form

$$\Psi_{\pm}(k) = S(k)^{-1} \left(\prod_{l=k_0}^{k-1} (\lambda_{\pm}(k) + \tilde{V}(l)_{\pm}) \right) (e_{\pm} + l_1^2)$$

where $\tilde{V}_+ = \tilde{V}_{11}$, $\tilde{V}_- = \tilde{V}_{22}$, $e_+ = e_1$, $e_- = e_2$, and $\|\tilde{V}(l)\| \in l_1^2$. We also have

$$\lambda_{\pm}(k) + \tilde{V}(k)_{\pm} = \lambda_{\pm}(k)(1 \pm r_{\pm}(k))$$

where

$$r_{\pm}(k) = \frac{a(k+1)(\lambda_{\pm}(k) - \lambda_{\pm}(k+1)) + \lambda_{\mp}(k)(a(k+1) - a(k)) - b(k)}{a(k+1)(\lambda_{\pm}(k+1) - \lambda_{\mp}(k+1))}.$$

We now claim that

$$\sum_{l=k}^{\infty} r_{\pm}(l) \in l_1^2,$$

so in particular we can subsume the $1 + r$ terms into the error to write

$$\Psi_{\pm}(k) = c_{\pm} \left(\prod_{l=k_0}^{k-1} \lambda_{\pm}(l) \right) (S(k)^{-1} e_{\pm} + l_1^2).$$

To see this is indeed the case, we make the following observations. First, $a(k) \rightarrow 1$, $\lambda(J), \kappa(J) \in l_1^2$, and $\lambda_+(k)$ and $\lambda_-(k)$ tend to different finite constants. In this way we see

$$\sum_{l=k}^{\infty} \left(\frac{\lambda_{\mp}(l)(a(l+1) - a(l)) - b(l)}{a(l+1)(\lambda_{\pm}(l+1) - \lambda_{\mp}(l+1))} \right) \in l_1^2.$$

Second, we can write $\lambda_+(k) - \lambda_+(k+1)$ as

$$\begin{aligned} & \frac{\sqrt{E^2 - 4a_{k+1}^2}}{2a_{k+1}} - \frac{\sqrt{E^2 - 4a_{k+2}^2}}{2a_{k+2}} = \frac{a_{k+2}\sqrt{E^2 - 4a_{k+1}^2} - a_{k+1}\sqrt{E^2 - 4a_{k+2}^2}}{2a_{k+1}a_{k+2}} \\ & = \sqrt{E^2 - 4a_{k+1}^2} \left(\frac{a_{k+1} - a_{k+2}}{2a_{k+1}a_{k+2}} \right) \\ & \quad + a_{k+1} \frac{\sqrt{E^2 - 4a_{k+2}^2} - \sqrt{E^2 - 4a_{k+1}^2}}{a_{k+1}a_{k+2}}. \end{aligned}$$

Because $\kappa(J) \in l_1^2$, the first term is summable to be in l_1^2 as well. To see the same is true of the second term, we do a Taylor expansion of $\sqrt{E^2 - 4a^2}$ around the point $E^2 - 4$. After cancelling the constant terms we see that because $\kappa(J) \in l_1^2$ we have

$$\sum_{k=n}^{\infty} a_{k+1} \left(\frac{\sqrt{E^2 - 4a_{k+2}^2} - \sqrt{E^2 - 4a_{k+1}^2}}{a_{k+1}a_{k+2}} \right) \in l_1^2.$$

So the second term sums to be in l_1^2 as well, proving the claim

Now, $E = \beta + \beta^{-1}$ and

$$\beta^{\pm 1} = \frac{E \pm \sqrt{E^2 - 4}}{2}$$

so

$$\begin{aligned}\lambda_{\pm}(k) &= \frac{\beta^{\pm 1}}{a(k+1)} \left(\frac{E \pm \sqrt{E^2 - 4a(k+1)^2}}{E \pm \sqrt{E^2 - 4}} \right) \\ &= \frac{\beta^{\pm 1}}{a(k+1)} (1 \pm q_{\pm}(k)).\end{aligned}$$

Arguing as we did for the r_{\pm} terms we find

$$\sum_{l=k}^{\infty} q_{\pm}(l) \in l_1^2,$$

so we can subsume these products into the error term as well. Finally, using that $\kappa(J) \in l_1^2$ and taking the top row of Ψ_{\pm} we see

$$\psi_{\pm}(k) = c_{\pm} \beta^{\pm k} (1 + l_1^2),$$

as claimed. \square

Proof of Proposition 1.4(1). Recall that

$$\begin{aligned}|\tilde{\kappa}(n-1) - \kappa(n-1)| &= \left| \sum_{k=n}^{\infty} a(k)^2 \left(\frac{c(k-1)c(k+1)}{c(k)^2} - 1 \right) \right| \\ |\tilde{\lambda}(n-1) - \lambda(n-1)| &= |\gamma| \left| \sum_{k=n}^{\infty} \left(\frac{a(k-1)\phi(k-1)\phi(k)}{c(k-1)} - \frac{a(k)\phi(k)\phi(k+1)}{c(k)} \right) \right|\end{aligned}$$

where $J\phi = E\phi$, $\phi(0) = 0$, $\phi(1) = 1$ and

$$c(n) = 1 + \gamma \sum_{j=1}^n |\phi(j)|^2.$$

Write ϕ as a linear combination of ψ_+ and ψ_- . Let us first suppose that ϕ is just a multiple of ψ_- . As ψ_- is geometrically decreasing, the same is true of

$$a(k)^2 \left(\frac{c(k-1)c(k+1)}{c(k)^2} - 1 \right)$$

and

$$\left(\frac{a(k-1)\phi(k-1)\phi(k)}{c(k-1)} - \frac{a(k)\phi(k)\phi(k+1)}{c(k)} \right).$$

So in this case it is easy to see that $|\tilde{\kappa}(n-1) - \kappa(n-1)|$ and $|\tilde{\lambda}(n-1) - \lambda(n-1)|$ are in l_1^2 .

Now suppose that ϕ is not just a multiple of ψ_- . As ψ_+ increases geometrically and ψ_- decays geometrically, we see

$$(3.2) \quad c(k) \sim 1 + \gamma \sum_{l=1}^k \psi(l)^2 \sim 1 + \gamma \sum_{l=1}^k \beta^{2l} (1 + \tilde{\delta}(l)) \sim 1 + \beta^{2k} (1 + \delta(k))$$

where $\tilde{\delta}(k), \delta(k)$ represent some sequences in l_1^2 , and “ \sim ” indicates asymptotic equivalence (modulo multiplication by constants). Similarly

$$\psi(k)\psi(k+1) \sim \beta^{2k+1} (1 + \varepsilon(k))$$

for some $\varepsilon \in l_1^2$. Combining these shows

$$\begin{aligned} & \left| \frac{a(k-1)\psi(k-1)\psi(k)}{c(k-1)} - \frac{a(k)\psi(k)\psi(k+1)}{c(k)} \right| \\ & \lesssim \left| \frac{a(k-1)\beta^{2k-1}(1+\varepsilon(k-1)) - a(k)\beta^{2k+1}(1+\varepsilon(k))}{c(k-1)c(k)} \right| \\ & \quad + \left| \frac{\beta^{4k-1}(a(k-1)(1+\varepsilon(k-1)) - a(k)(1+\varepsilon(k)))}{c(k-1)c(k)} \right| \\ & \quad + \left| \frac{\beta^{4k-1}(a(k-1)\delta(k)(1+\varepsilon(k-1)) - a(k)\delta(k-1)(1+\varepsilon(k)))}{c(k-1)c(k)} \right|. \end{aligned}$$

Because $c(k-1)c(k) \sim \beta^{4k-1}$, the first term is geometrically decreasing, so okay by Lemma 2.4.

Terms of the form

$$\frac{a(k-1)\beta^{4k-1}}{c(k-1)c(k)}\varepsilon(k-1)\delta(k)$$

are in l_1^1 , being products of l_1^2 sequences. Again, Lemma 2.4 shows this is fine.

This leaves terms of the form

$$\frac{\beta^{4k-1}}{c(k-1)c(k)}(\varepsilon(k-1) - \varepsilon(k))$$

for some sequence $\varepsilon \in l_1^2$. So it is sufficient to prove

$$\sum_{k=n}^{\infty} \left(\frac{\beta^{4k-1}}{c(k-1)c(k)}(\varepsilon(k-1) - \varepsilon(k)) \right) \in l_1^2.$$

Let

$$C(k) = \frac{\beta^{4k-1}}{c(k-1)c(k)}.$$

Summing by parts shows

$$(3.3) \quad \sum_{k=n}^{\infty} C(k)(\varepsilon(k-1) - \varepsilon(k)) = C(n)\varepsilon(n-1) + \sum_{k=n}^{\infty} \varepsilon(k)(C(k+1) - C(k)).$$

The first term is clearly in l_1^2 , so consider the second. Using (3.2) we can write

$$\begin{aligned} & |C(k+1) - C(k)| \\ & = \left| \frac{\beta^{4k+3}}{c(k)c(k+1)} - \frac{\beta^{4k-1}}{c(k)c(k+1)} \right| \\ & \lesssim \left| \frac{\beta^{4k-1}}{c(k-1)c(k)c(k+1)} \left((\beta^4 - 1) + \beta^{2k+2}(\delta(k-1) - \delta(k+1)) \right) \right|. \end{aligned}$$

As $c(k-1)c(k)c(k+1) \sim \beta^{6k}$ the first term is geometrically decaying and the second term is in l_1^2 . Combining this with (3.3) and Lemma 2.4 shows that

$$\sum_{k=n}^{\infty} \left(\frac{\beta^{4k-1}}{c(k-1)c(k)}(\varepsilon(k-1) - \varepsilon(k)) \right) \in l_1^2.$$

This completes the proof for the λ 's. The proof for the κ 's is similar and simpler. \square

4. PROOF OF THEOREM 1.1 ((2) \Rightarrow (1))

By assumption, J has finitely many eigenvalues, and they all lie in $\mathbb{R} \setminus [-2, 2]$. By Theorem 1.3 and Proposition 1.4 we see it suffices to prove the theorem when $\sigma(J) \subseteq [-2, 2]$, which we now assume.

Now, $d\nu(x) = \chi_{[-2, 2]}(x)v(x)dx$ and $v(x)$ has one of the forms

$$\left(\sqrt{4-x^2}\right)^{\pm 1} v_0(x) \quad \text{or} \quad \left(\sqrt{\frac{2-x}{2+x}}\right)^{\pm 1} v_0(x)$$

with $\log v_0 \in \dot{H}^{1/2}$. Define

$$w(\theta) = cv_0(2\cos\theta)$$

with c chosen to normalize w to be a probability measure on $\partial\mathbb{D}$. Notice that

$$d\nu = Sz^{(*)}(d\mu)$$

where $d\mu = w \frac{d\theta}{2\pi}$ and $(*)$ is one of $(e), (o), (+), (-)$ according to which of the above forms $v(x)$ has. Therefore, α and a, b are related by one of the Direct Geronimus Relations of Proposition 1.5.

By assumption $\log w \in \dot{H}^{1/2}$, so Theorem 1.2 shows that its Verblunsky coefficients satisfy $\alpha \in l_1^2$. By Proposition 1.5 we see that

$$\begin{aligned} \kappa_n &= \alpha_{2n-1} + K(\alpha)_n \\ \lambda_n &= \alpha_{2n-2} + L(\alpha)_n \end{aligned}$$

where $K(\alpha)_n$ and $L(\alpha)_n$ are sums from n to infinity of terms that are at least quadratic in α . So by Lemma 2.4 we see that $\lambda, \kappa \in l_1^2$ too.

5. ASYMPTOTIC INTEGRATION REDUX

For this section we will make the standing assumption that the parameters defined by (1.1) obey

$$\kappa(J), \lambda(K) \in l_1^2.$$

In Section 7 we will need asymptotics on solutions at energies $E = \pm 2$. As before we will use asymptotic integration, but because the recurrence equation at $E = \pm 2$ yields a system with a Jordan anomaly, we cannot use the results of Section 2. Instead we construct a small solution ψ_s and big solution ψ_b :

Proposition 5.1. *There are solutions ψ_s and ψ_b to $J\psi = E\psi$ at energy $E = \pm 2$ such that*

$$\left| \frac{\psi_s(k)}{\psi_b(k)} \right| \rightarrow 0$$

and

$$\pm \frac{\psi_s(k+1)}{\psi_s(k)} = 1 + l_1^2.$$

Moreover, for either solution and for k sufficiently large,

$$(\pm 1)^k \psi(k) > 0.$$

The rest of this section is devoted to a proof of this statement for $E = 2$, the proof for $E = -2$ being analogous. Recall we can write the recurrence equation as

$$\Psi(k+1) = \begin{bmatrix} \frac{2-b(k+1)}{a(k+1)} & -\frac{a(k)}{a(k+1)} \\ 1 & 0 \end{bmatrix} \Psi(k)$$

where

$$\Psi(k) = \begin{bmatrix} \psi(k) \\ \psi(k-1) \end{bmatrix}.$$

We begin with some preliminary transformations. Let

$$S = \begin{bmatrix} 1 & 1/2 \\ 1 & -1/2 \end{bmatrix}$$

and let $\Phi(k) = S\Psi(k)$. Then Φ solves

$$\Phi(k+1) = [J + B(k)]\Phi(k)$$

where

$$J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$B(k) = \begin{bmatrix} \frac{(a(k+1)-1) + (a(k)-1) - b(k)}{2a(k+1)} & \frac{-3(a(k+1)-1) + (a(k)-1) - b(k)}{4a(k+1)} \\ \frac{2(a(k+1)-1) + (a(k)-1) - b(k)}{a(k+1)} & \frac{-3(a(k+1)-1) + (a(k)-1) - b(k)}{2a(k+1)} \end{bmatrix}.$$

In particular, notice that

$$\left\| \sum_{l=k}^{\infty} B(l) \right\| \in l_1^2.$$

Lemma 5.2. *There exists a sequence of matrices*

$$Q(k) = \begin{bmatrix} 1 & 1 \\ q(k) & 1 \end{bmatrix}$$

such that $q \in l_1^2$,

$$(5.1) \quad Q(k+1)^{-1}[J + B(k)]Q(k) = L(k) + M(k),$$

and

$$\|M(k)\| \in l_1^1, \quad L(k) = \begin{bmatrix} 1 + \alpha(k) & 1 + \beta(k) \\ 0 & 1 + \gamma(k) \end{bmatrix},$$

where

$$\sum_{l=k}^{\infty} \alpha(l) \in l_1^2, \quad \sum_{l=k}^{\infty} \beta(l) \in l_1^2, \quad \gamma(k) \in l_1^2.$$

In particular, if $\Phi(k) = Q(k)x(k)$ then

$$x(k+1) = [L(k) + M(k)]x(k).$$

Proof. Define

$$Q(k) = \begin{bmatrix} 1 & 1 \\ q(k) & 1 \end{bmatrix}$$

where $q(k) = -\sum_{l=k}^{\infty} B(l)_{21} \in l_1^2$ by Lemma 2.4. All the claimed properties are straightforward calculations. \square

As we seek asymptotics as $k \rightarrow \infty$, we need only consider systems for k larger than some k_0 . In particular, we can choose k_0 so that $|\alpha(k)|, |\beta(k)|, |\gamma(k)| < 1$ for $k \geq k_0$. In this case define

$$x(k) = P(k)z(k)$$

where

$$P(k) = \begin{bmatrix} 1 & 0 \\ 0 & \prod_{j=1}^{k-1} (1 + \gamma(j)) \end{bmatrix}.$$

This transforms the x -system into

$$(5.2) \quad z(k+1) = [J(k) + R(k)]z(k)$$

where

$$J(k) = \begin{bmatrix} 1 + \alpha(k) & 1 + \beta(k) \\ 0 & 1 \end{bmatrix}$$

and $\|R(k)\| \in l_1^1$. We will compare this to the simpler system

$$y(k+1) = J(k)y(k).$$

We begin by finding a basis of solutions to the y -system.

Lemma 5.3. *The y -system above has two solutions*

$$y_s(k) = \begin{bmatrix} u(k) \\ 0 \end{bmatrix} \quad \text{and} \quad y_b(k) = \begin{bmatrix} v(k) \\ 1 \end{bmatrix}$$

such that

$$u(k) = \prod_{j=1}^{k-1} (1 + \alpha(j))$$

$$(5.3) \quad 0 < |u(k)| \lesssim 1$$

$$|v(k)| \sim k.$$

Proof. Let $u(1) = 1$ and $v(1) = 0$. By the form of $J(k)$ we see that

$$u(k) = \prod_{j=1}^{k-1} (1 + \alpha(j)).$$

Because the α 's are conditionally summable, the product defining $u(k)$ converges to some finite number as $k \rightarrow \infty$, so $|u(k)| \lesssim 1$. As we have assumed that $|\alpha(k)| < 1$, we also have $u(k) \neq 0$.

Now,

$$(5.4) \quad \begin{aligned} v(k+1) &= (1 + \alpha(k))v(k) + (1 + \beta(k)) \\ &= \sum_{j=1}^k \frac{u(k+1)}{u(j+1)} (1 + \beta(j)). \end{aligned}$$

By (5.3) and $\beta(j) \rightarrow 0$ we have

$$|v(k)| \lesssim \sum_{l=1}^k 1 \lesssim k.$$

Moreover, there is some j_0 so that for $k \geq j \geq j_0$,

$$\frac{u(k+1)}{u(j+1)} \left(1 + \beta(j)\right)$$

is sign-definite. Without loss, assume that it is positive, so for j and k large enough we have

$$\frac{u(k+1)}{u(j+1)} \left(1 + \beta(j)\right) \gtrsim 1.$$

Thus, $|v(k)| \gtrsim k$ too. □

Now let

$$Y(k) = \begin{bmatrix} u(k) & v(k) \\ 0 & 1 \end{bmatrix}$$

be a fundamental matrix for the y -system. The next two lemmas construct the desired solutions to (5.2).

Lemma 5.4. *There is a bounded solution to the system (5.2) that has*

$$\|z(k+1) - z(k)\| \in l_1^2.$$

Moreover, $z(k)$ is sign-definite for large enough k , and $\|z(k)\| > 0$.

Proof. Consider the operator

$$(5.5) \quad [Tz](k) = - \sum_{l=k}^{\infty} Y(k)Y(l+1)^{-1}R(l)z(l)$$

acting on $l^\infty(\mathbb{N}; \mathbb{C}^2)$, with $k \geq k_1$ and k_1 to be chosen momentarily. Notice that

$$Y(k)Y(l+1)^{-1} = \begin{bmatrix} \frac{u(k)}{u(l+1)} & v(k) - \frac{u(k)}{u(l+1)}v(l+1) \\ 0 & 1 \end{bmatrix}.$$

By Lemma 5.3,

$$\left| \frac{u(k)}{u(l+1)} \right| \lesssim 1.$$

By (5.4) we see that

$$v(k) - \frac{u(k)}{u(l+1)}v(l+1) = - \sum_{j=k}^l \frac{u(k)}{u(j+1)} \left(1 + \beta(j)\right)$$

so

$$\left| v(k) - \frac{u(k)}{u(l+1)}v(l+1) \right| \lesssim |l - k|.$$

Thus

$$(5.6) \quad \|Y(k)Y(l+1)^{-1}\| \lesssim |l - k|.$$

If $z \in l^\infty$ we see

$$\|[Tz](k)\|_\infty \lesssim \|z\|_\infty \sum_{l=k_1}^{\infty} l \|R(l)\|.$$

Now, $\|R(l)\| \in l_1^1$, so by choosing k_1 sufficiently large, we can ensure

$$\|Tz\| < \varepsilon \|z\|$$

for some $\varepsilon < 1$. Thus, T is a contraction on l^∞ , so in particular, given any $y \in l^\infty$ there is a unique $z \in l^\infty$ solving

$$z = y + Tz.$$

If $y = y_s$ from Lemma 5.3, then by the form of T (and a lengthy but easy calculation) we see that this z solves the z -equation.

Since

$$\|Tz\| < \varepsilon\|z\|$$

we see that

$$\|y_s\| \leq \|z\| + \|Tz\| < (1 + \varepsilon)\|z\|$$

so

$$\|z\| \gtrsim \|y_s\| > 0$$

by Lemma 5.3. Moreover, because $y_s(k)$ is sign-definite for large enough k and $\|Tz\| < \varepsilon\|z\|$, we see the same is true of z .

Next, notice that

$$\|z(k+1) - z(k)\| \leq \|y_s(k+1) - y_s(k)\| + \|[Tz](k+1) - [Tz](k)\|.$$

For the first term we use Lemma 5.3 to see

$$\|y_s(k+1) - y_s(k)\| \lesssim |\alpha(k)| \in l_1^2.$$

For the second term we use (5.5) to write

$$(5.7) \quad [Tz](k+1) - [Tz](k) \\ = Y(k)Y(k+1)^{-1}R(k)z(k) - \sum_{l=k+1}^{\infty} [Y(k+1) - Y(k)]Y(l+1)^{-1}R(l)z(l).$$

Now,

$$[Y(k+1) - Y(k)]Y(l+1)^{-1} = \\ \begin{bmatrix} \frac{u(k+1)-u(k)}{u(l+1)} & \left(v(k+1) - \frac{u(k+1)}{u(l+1)}v(l+1)\right) + \left(v(k) - \frac{u(k)}{u(l+1)}v(l+1)\right) \\ 0 & 0 \end{bmatrix}.$$

By Lemma 5.3,

$$\left| \frac{u(k+1) - u(k)}{u(l+1)} \right| \lesssim 1.$$

For the other term in the matrix we use (5.4) to rewrite

$$\begin{aligned} & \left(v(k+1) - \frac{u(k+1)}{u(j+1)}v(l+1)\right) + \left(v(k) - \frac{u(k)}{u(j+1)}v(j+1)\right) \\ &= - \sum_{j=k+1}^l \frac{u(k+1)}{u(j+1)}(1 + \beta(j)) + \sum_{j=k}^l \frac{u(k)}{u(j+1)}(1 + \beta(j)) \\ &= \frac{u(k)}{u(k+1)}(1 + \beta(k)) + (u(k) - u(k+1)) \sum_{j=k+1}^l \frac{1}{u(j+1)}(1 + \beta(j)) \\ &= \frac{u(k)}{u(k+1)}(1 + \beta(k)) - \alpha(k)u(k) \sum_{j=k+1}^l \frac{1}{u(j+1)}(1 + \beta(j)). \end{aligned}$$

In particular we have $\|[Y(k+1) - Y(k)]Y(l+1)^{-1}\| \lesssim 1 + |\alpha(k)|l$.

Plugging this into (5.7) and using $z \in l^\infty$ and (5.6) we find

$$\begin{aligned} \|[Tz](k+1) - [Tz](k)\| &\lesssim \|R(k)\| + \sum_{l=k+1}^{\infty} \|R(l)\| + |\alpha(k)| \sum_{l=k+1}^{\infty} l \|R(l)\| \\ &\lesssim \|R(k)\| + \sum_{l=k+1}^{\infty} \|R(l)\| + |\alpha(k)|. \end{aligned}$$

The first and third terms are clearly l_1^2 , and by Lemma 2.4 so is the second. Thus $\|z(k+1) - z(k)\| \in l_1^2$. \square

Lemma 5.5. *There is a solution*

$$z_b(k) = \begin{bmatrix} z_{b1}(k) \\ z_{b2}(k) \end{bmatrix}$$

to the z -system that is sign-definite for k large enough and has $|z_{b1}(k)| \sim k$ and $|z_{b2}(k)| \lesssim 1$.

Proof. Again, we compare the z -system to the y -system and use Lemma 5.3. This time, consider the operator

$$(5.8) \quad [Tz](k) = \sum_{l=k_1}^{k-1} Y(k)Y(l+1)^{-1}R(l)z(l)$$

with $k_1 \geq 1$ to be chosen momentarily. Let $z_0 = y_b$ and $z_{j+1} = y_b + Tz_j$. Then

$$\|z_{j+1}(k) - z_j(k)\| = \|T[z_j - z_{j-1}](k)\|$$

and

$$\|z_1(k) - z_0(k)\| = \|T[y](k)\|.$$

By (5.6) and Lemma 5.3 we have

$$\begin{aligned} \|[Ty](k)\| &\leq \sum_{l=k_1}^{k-1} \|Y(k)Y(l+1)^{-1}\| \cdot \|R(l)\| \cdot \|y(l)\| \\ &\lesssim k \sum_{l=k_1}^{k-1} \|R(l)\| l. \end{aligned}$$

We can choose k_1 sufficiently large that

$$\|[Ty](k)\| < k\varepsilon$$

where $\varepsilon < 1$. Then inductively we find that

$$\|z_{j+1}(k) - z_j(k)\| < k\varepsilon^{j+1}.$$

In particular, for each k , $z_j(k) \rightarrow z_b(k)$ as $j \rightarrow \infty$ and $z_b = y_b + Tz_b$.

By the form of T we see that because y_b solves the y -equation, z_b solves the z -equation. Moreover, as

$$\|[Tz_b](k)\| < k\varepsilon$$

and

$$\|y_b(k)\| \sim k$$

we have

$$(5.9) \quad \|z_b\| \sim k.$$

Finally, because $y_b(k)$ is sign-definite for large enough k and $\|[Tz_b](k)\| < k\varepsilon$, we see the same is true for z_b .

To deduce the component bounds, we expand

$$Y(k)Y(l+1)^{-1}R(l)z_b(l)$$

and notice that the bottom component is bounded by

$$|z_{b1}(l)R(l)_{21}| + |z_{b2}(l)R(l)_{22}|.$$

Plugging this into (5.8) shows

$$\begin{aligned} |z_{b2}(k)| &\lesssim \sum_{l=k_1}^{k-1} |z_{b1}(l)R(l)_{21}| + |z_{b2}(l)R(l)_{22}| \\ &\lesssim \sum_{l=k_1}^{k-1} l\|R(l)\| \lesssim 1. \end{aligned}$$

Combining this with (5.9) yields the final bound. \square

Proof of Proposition 5.1. Undoing the transformations we find that

$$\Psi(k) = SQ(k)P(k)z(k)$$

and therefore that

$$\psi(k) = \frac{1}{2} \left((1 + q(k))z_1(k) + 2 \prod_{j=1}^{k-1} (1 + \gamma(j))z_2(k) \right)$$

where

$$z(k) = \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix}.$$

Let ψ_s and ψ_b correspond to taking z to be z_s and z_b . All the claimed properties now follow from Lemmas 5.4 and 5.5. \square

6. THE INVERSE GERONIMUS RELATIONS

Recall that the Direct Geronimus Relations provide formulas for $a^{(*)}, b^{(*)}$ in terms of α . In this section we go the other way. We begin by determining whether a particular Jacobi matrix is in the range of the Szegő maps based on the values of its m -function. Note that while $Sz^{(e)}$ maps onto all probability measures supported on $[-2, 2]$, the ranges of the other three maps are given by

$$\begin{aligned} \text{Ran}(Sz^{(o)}) &= \left\{ d\nu : \int_{-2}^2 \frac{d\nu(x)}{4-x^2} < \infty \right\} \\ \text{Ran}(Sz^{(\pm)}) &= \left\{ d\nu : \int_{-2}^2 \frac{d\nu(x)}{2 \mp x} < \infty \right\}. \end{aligned} \tag{6.1}$$

If $x \in \mathbb{R}$, write

$$m(x + i0) = \lim_{\varepsilon \downarrow 0} m(x + i\varepsilon),$$

and write $m(x)$ to indicate the value of the integral

$$\int \frac{d\nu(x)}{x - z}$$

(which may be infinite).

We begin by developing some elementary properties of the m -functions, which we then use to study the associated polynomials.

Lemma 6.1. *Let J be a Jacobi matrix with $\sigma(J) \subseteq [-2, 2]$. Then*

$$J \in \text{Ran}(Sz^{(o)}) \Leftrightarrow m(-2) - m(2) < \infty$$

$$J \in \text{Ran}(Sz^{(\pm)}) \Leftrightarrow \mp m(\pm 2) < \infty.$$

Proof. The second line follows from the definition of the m -function and (6.1). For the first line note

$$m(-2) - m(2) = \int_{-2}^2 \left(\frac{1}{2+x} + \frac{1}{2-x} \right) d\nu(x) = 4 \int_{-2}^2 \frac{d\nu(x)}{4-x^2}$$

and again use (6.1). \square

As with the ranges, the normalization constants (1.5) have interpretations in terms of the m -function:

Lemma 6.2. *If $m^{(*)}(x)$ is the m -function for $d\nu^{(*)}$ then*

$$\mp m^{(o)}(\pm 2) = \frac{1}{(1 \mp \alpha_0)(1 - \alpha_1)}$$

$$\mp m^{(\pm)}(\pm 2) = \frac{1}{2(1 \mp \alpha_0)}.$$

Proof. By (1.6) we can write

$$d\nu^{(o)}(x) = \frac{(2-x)(2+x)}{2(1-\alpha_0^2)(1-\alpha_1)} d\nu^{(e)}(x)$$

$$d\nu^{(\pm)}(x) = \frac{2 \mp x}{2(1 \mp \alpha_0)} d\nu^{(e)}(x).$$

The values of $m^{(\pm)}$ then follow from $d\nu^{(e)}$ being a probability measure. For the $m^{(o)}$ values we have

$$\begin{aligned} m^{(o)}(-2) &= \int_{-2}^2 \frac{d\nu^{(o)}(x)}{2+x} = \frac{1}{2(1-\alpha_0^2)(1-\alpha_1)} \int_{-2}^2 (2-x) d\nu^{(e)}(x) \\ &= \frac{1}{2(1-\alpha_0^2)(1-\alpha_1)} \int_0^{2\pi} 2 - (z + z^{-1}) d\mu(z) \\ &= \frac{1}{(1-\alpha_0^2)(1-\alpha_1)} \left(1 - \int_0^{2\pi} z d\mu(z) \right) \\ &= \frac{1-\alpha_0}{2(1-\alpha_0^2)(1-\alpha_1)}. \end{aligned}$$

The value of $-m^{(o)}(2)$ follows similarly. \square

We'll need lower bounds on the m -function:

Lemma 6.3. *If $\sigma(J) \subseteq [-2, 2]$, then $\mp m(\pm 2) > 1/4$.*

Proof. As J has no eigenvalues off $[-2, 2]$,

$$m(E) = \int_{-2}^2 \frac{d\nu(x)}{x-E}.$$

For $t \in [-2, 2]$ and $E > 2$, $t - E \geq -4$. Because $d\nu$ is a probability measure that is not a point mass at $t = 2$, the Monotone Convergence Theorem implies

$$-m(2) = \lim_{E \downarrow 2} -m(E) > 1/4.$$

Similar arguments show $m(-2) > 1/4$. \square

We now turn to the polynomials. Given $d\mu$, write $P_n^{(*)}(x)$ for the monic polynomial of degree n with respect to the measure $d\nu^{(*)} = Sz^{(*)}(d\mu)$. Similarly, let $Q_n^{(*)}(x)$ be the second-kind polynomial for $d\nu^{(*)}$. That is, Q solves the same recurrence equation as P but with initial conditions $Q_{-1} \equiv -1$ and $Q_0 \equiv 0$. If $|m(x)| < \infty$, let $F_n^{(*)}(x) = m(x)P_n^{(*)}(x) + Q_n^{(*)}(x)$.

Proposition 6.4. *Let $d\mu$ a nontrivial probability measure on $\partial\mathbb{D}$ that is invariant under conjugation, and let α be its Verblunsky parameters. Then*

$$\begin{aligned} P_{n+1}^{(e)}(2) &= (1 - \alpha_{2n-1})(1 - \alpha_{2n})P_n^{(e)}(2) \\ P_{n+1}^{(e)}(-2) &= -(1 - \alpha_{2n-1})(1 + \alpha_{2n})P_n^{(e)}(-2) \\ F_{n+1}^{(o)}(2) &= (1 + \alpha_{2n+1})(1 + \alpha_{2n+2})F_n^{(o)}(2) \\ F_{n+1}^{(o)}(-2) &= -(1 + \alpha_{2n+1})(1 - \alpha_{2n+2})F_n^{(o)}(-2) \\ F_{n+1}^{(+)}(2) &= (1 + \alpha_{2n})(1 + \alpha_{2n+1})F_n^{(+)}(2) \\ P_{n+1}^{(+)}(-2) &= -(1 + \alpha_{2n})(1 - \alpha_{2n+1})P_n^{(+)}(-2) \\ P_{n+1}^{(-)}(2) &= (1 - \alpha_{2n})(1 - \alpha_{2n+1})P_n^{(-)}(2) \\ F_{n+1}^{(-)}(-2) &= -(1 - \alpha_{2n})(1 + \alpha_{2n+1})F_n^{(-)}(-2). \end{aligned}$$

Proof. The proof is by induction. As the arguments for any of the P 's are virtually identical, we only present the proof for the case $P = P^{(e)}$. Similarly, we only present the argument for the F 's in the case $F = F^{(+)}$.

The desired relationship between $P_0 \equiv 1$ and $P_1(x) = x - b_1$ follows from Proposition 1.5:

$$P_1(2) = 2 - b_1 = 2 - (\alpha_0(1 - \alpha_{-1})) = 2(1 - \alpha_0) = (1 - \alpha_{-1})(1 - \alpha_0)P_0(2).$$

To deduce the desired relationship between $F_1(2)$ and $F_0(2)$, we argue as follows. By Lemma 6.3 and Lemma 6.1 we have that $\frac{1}{4} < -m(2) < \infty$, so $F_0(2) = m(2)$. Next, recall that $P_{-1} \equiv 0$, $P_0 \equiv 1$, $Q_{-1} \equiv -1$, and $Q_0 \equiv 0$. So

$$\begin{aligned} F_1(2) &= m(2)P_1(2) + Q_1(2) \\ &= -\frac{(2 - b_1)}{2(1 - \alpha_0)} + 1 = -\frac{2\alpha_0 - b_1}{2(1 - \alpha_0)} \\ &= (1 + \alpha_0)(1 + \alpha_1)\frac{-1}{2(1 - \alpha_0)} = (1 + \alpha_0)(1 + \alpha_1)F_0(2) \end{aligned}$$

where we have used Proposition 1.5 and Lemma 6.2.

Now, assume the formulas hold up to index $n-1$. As P_n satisfies the three-term recurrence equation we have

$$\begin{aligned} P_{n+1}(2) &= (2 - b_{n+1})P_n(2) - a_n^2 P_{n-1}(2) \\ &= \left((2 - b_{n+1}) - \frac{a_n^2}{(1 - \alpha_{2n-3})(1 - \alpha_{2n-2})} \right) P_n(2) \\ &= (1 - \alpha_{2n-1})(1 - \alpha_{2n})P_n(2) \end{aligned}$$

where the second equality is by the inductive hypothesis, and the third equality is by Proposition 1.5.

Similarly, F_n satisfies the three-term recurrence equation, so the same argument works:

$$\begin{aligned} F_{n+1}(2) &= (2 - b_{n+1})F_n(2) - a_n^2 F_{n-1}(2) \\ &= \left((2 - b_{n+1}) - \frac{a_n^2}{(1 + \alpha_{2n-2})(1 + \alpha_{2n-1})} \right) F_n(2) \\ &= (1 + \alpha_{2n})(1 + \alpha_{2n+1})F_n(2). \end{aligned}$$

□

Proposition 6.5 (Inverse Geronimus Relations⁵). *Let $d\mu$ a nontrivial probability measure on $\partial\mathbb{D}$ that is invariant under conjugation, and let α be its Verblunsky parameters. Define*

$$\begin{aligned} A_n^{(*)} &= -\frac{P_{n+1}^{(*)}(-2)}{P_n^{(*)}(-2)} & B_n^{(*)} &= \frac{P_{n+1}^{(*)}(2)}{P_n^{(*)}(2)} \\ C_n^{(*)} &= -\frac{F_{n+1}^{(*)}(-2)}{F_n^{(*)}(-2)} & D_n^{(*)} &= \frac{F_{n+1}^{(*)}(2)}{F_n^{(*)}(2)}. \end{aligned}$$

If $d\nu = Sz^{(e)}(d\mu)$

$$\alpha_{2n} = \frac{A_n^{(e)} - B_n^{(e)}}{A_n^{(e)} + B_n^{(e)}} \quad \alpha_{2n-1} = 1 - \frac{1}{2}(A_n^{(e)} + B_n^{(e)}).$$

If $d\nu = Sz^{(o)}(d\mu)$

$$-\alpha_{2n+2} = \frac{C_n^{(o)} - D_n^{(o)}}{C_n^{(o)} + D_n^{(o)}} \quad -\alpha_{2n+1} = 1 - \frac{1}{2}(C_n^{(o)} + D_n^{(o)}).$$

If $d\nu = Sz^{(+)}(d\mu)$

$$-\alpha_{2n+1} = \frac{A_n^{(+)} - D_n^{(+)}}{A_n^{(+)} + D_n^{(+)}} \quad -\alpha_{2n} = 1 - \frac{1}{2}(A_n^{(+)} + D_n^{(+)}).$$

If $d\nu = Sz^{(-)}(d\mu)$

$$\alpha_{2n+1} = \frac{C_n^{(-)} - B_n^{(-)}}{C_n^{(-)} + B_n^{(-)}} \quad \alpha_{2n} = 1 - \frac{1}{2}(C_n^{(-)} + B_n^{(-)}).$$

⁵The case $d\nu = Sz^{(e)}(d\mu)$ is due to [4] (with an alternate proof given in [3]). The statement in the other three cases appears to be new (although anticipated in [16] and related to some formulas of [2]).

By Sturm oscillation theory and that $Sz^{(*)}(d\mu)$ is supported in $[-2, 2]$, we see $(\pm 1)^{n+1}P_n^{(*)}(\pm 2)$ and $-(\pm 1)^{n+1}F_n^{(*)}(\pm 2)$ are strictly positive for all $n > 0$. In particular, the above ratios are all defined.

Proof. This is a simple calculation based on Proposition 6.4. \square

7. SOME WEYL THEORY

By Proposition 6.5 we see that decay of the Verblunsky parameters is controlled by decay of the sequences A_n , B_n , C_n , and D_n . By Proposition 5.1 we see that there is a solution at $E = \pm 2$ with the desired asymptotics. The following result connects these two ideas.

Proposition 7.1. *Let J be a Jacobi matrix with $\sigma(J) \subseteq [-2, 2]$, and let A_n , B_n , C_n , D_n be defined as above. Then $m(-2) < \infty$ implies $C_n = 1 + l_1^2$, and $m(-2) = \infty$ implies $A_n = 1 + l_1^2$. Similarly, $-m(2) < \infty$ implies $D_n = 1 + l_1^2$, and $-m(2) = \infty$ implies $B_n = 1 + l_1^2$.*

Let us write p_n and q_n for the orthonormal versions of P_n and Q_n , and then $f_n(z) = m(z)p_n(z) + q_n(z)$. Proposition 7.1 is a trivial consequence of

Proposition 7.2. *Let J be a Jacobi matrix with $\sigma(J) \subseteq [-2, 2]$. Then $m(-2) < \infty$ implies $(-1)^{n+1}f_n(-2) = s + l_1^2$, and $m(-2) = \infty$ implies $(-1)^{n+1}p_n(-2) = s + l_1^2$, for some $s \in \mathbb{R}$. Similarly, $-m(2) < \infty$ implies $f_n(2) = s + l_1^2$, and $-m(2) = \infty$ implies $p_n(2) = s + l_1^2$.*

To prove this, we will use some Weyl theory. Recall $p_n(z)$ and $q_n(z)$ are solutions to $Ju = zu$ with $p_{-1} = q_0 = 0$ and $p_0 = -q_{-1} = 1$. When $z \in \mathbb{C} \setminus \mathbb{R}$, the Weyl solution $f_n(z) = m(z)p_n(z) + q_n(z)$ is defined and satisfies

$$(7.1) \quad \|f_n(z)\|_{l^2}^2 = \frac{\operatorname{Im} m(z)}{\operatorname{Im} z}.$$

As the m -function and the solutions p and q will play prominent roles, we develop some of their key properties. To start, we relate the values of m at ± 2 to its values at $\pm 2 + i\varepsilon$.

Lemma 7.3. *Let J be a Jacobi matrix with $\sigma(J) \subseteq [-2, 2]$, m -function m , and spectral measure $d\nu$. Then*

$$\begin{aligned} \int_{-2}^2 \frac{d\nu(t)}{2+t} < \infty &\Rightarrow m(-2) = m(-2 + i0) \\ \int_{-2}^2 \frac{d\nu(t)}{2+t} = \infty &\Rightarrow |m(-2 + i0)| = \infty. \end{aligned}$$

Similarly,

$$\begin{aligned} \int_{-2}^2 \frac{d\nu(t)}{2-t} < \infty &\Rightarrow m(2) = m(2 + i0). \\ \int_{-2}^2 \frac{d\nu(t)}{2-t} = \infty &\Rightarrow |m(2 + i0)| = \infty. \end{aligned}$$

In particular, when $m(\pm 2)$ is finite, we may write $m(\pm 2)$ for $m(\pm 2 + i0)$ and then $f_n(\pm 2)$ for $f_n(\pm 2 + i0)$.

Notice that by Lemma 6.3, $\mp m(\pm 2)$ can only diverge to $+\infty$.

Proof. The first implication follows from the Dominated Convergence Theorem applied to

$$-m(2 + i\varepsilon) = \int_{-2}^2 \frac{v(t)dt}{(2-t) + i\varepsilon}.$$

The second implication follows from the Monotone Convergence Theorem applied to

$$-\operatorname{Re} m(2 + i\varepsilon) = \int_{-2}^2 \frac{2-t}{(2-t)^2 + \varepsilon^2} d\nu(t).$$

□

If $L \in \mathbb{N}$ and $u(n; z)$ solves $Ju = zu$, we define

$$\|u(z)\|_L^2 = \sum_{l=0}^L |u(l; z)|^2.$$

For non-integer values of L we define $\|u(z)\|_L$ to be the linear interpolation between the values at $\lfloor L \rfloor$ and $\lceil L \rceil$. Now suppose $x \in \mathbb{R}$ is fixed and $m(x + i0)$ exists finitely. Let $\varepsilon, y' > 0$ be related by

$$\sup_{0 < y \leq y'} |m(z) - m(x + i0)| + y' = \frac{\varepsilon^2}{4}$$

where $z = x + iy$. Note that y' is a monotone function of ε that goes to zero as ε goes to zero. Define $L(\varepsilon)$ by

$$|y'|^{1/2} \|p_n(z')\|_{L(\varepsilon)} = 1$$

where $z' = x + iy'$. For each $y' > 0$, $L(\varepsilon)$ exists because $p_n(z')$ is not in l^2 .

The following lemma is a discrete analog of Lemma 9 of [6]. The proof is a direct translation, so we omit it.

Lemma 7.4. *Let $x \in \mathbb{R}$ and suppose that $m(x + i0)$ exists finitely. Then*

$$\frac{\|f_n(x + i0)\|_{L(\varepsilon)}}{\|p_n(x)\|_{L(\varepsilon)}} < \varepsilon$$

whenever ε is sufficiently small.

Next we recall a result of [12].

Lemma 7.5. *Let $x \in \mathbb{R}$ and define $\tilde{L}(\varepsilon)$ by*

$$\|p_n(x)\|_{\tilde{L}(\varepsilon)} \|q_n(x)\|_{\tilde{L}(\varepsilon)} = \frac{1}{2\varepsilon}.$$

Then $\tilde{L}(\varepsilon)$ is a well defined, monotonely decreasing continuous function that goes to infinity as ε goes to 0, and

$$\frac{5 - \sqrt{24}}{|m(x + i\varepsilon)|} \leq \frac{\|p_n(x)\|_{\tilde{L}(\varepsilon)}}{\|q_n(x)\|_{\tilde{L}(\varepsilon)}} \leq \frac{5 + \sqrt{24}}{|m(x + i\varepsilon)|}.$$

Proof of Proposition 7.2. Again, we will only prove the statements for $E = 2$. Suppose first that $1/4 \leq -m(2) < \infty$. Then by Lemma 7.3, $m(2 + i0)$ is finite and nonzero too. So by Lemma 7.5 we have that

$$\frac{\|p_n(2)\|_L}{\|q_n(2)\|_L}$$

remains finite and nonzero as $L \uparrow \infty$. As solutions at $E = 2$ are of the form $c_1\psi_b + c_2\psi_s$ for some $c_i \in \mathbb{R}$, we see that $p_n(2)$ and $q_n(2)$ must be simultaneously bounded or simultaneously unbounded. Because $p_n(2)$ and $q_n(2)$ form a basis for solutions at $E = 2$, we see they cannot both be bounded. So they are both unbounded.

Now, by Lemma 7.4 we see that

$$\frac{\|f_n(2)\|_L}{\|p_n(2)\|_L} \rightarrow 0$$

as $L \uparrow \infty$. So $f_n(2)$ cannot be unbounded, and so has the form $c\psi_s$ for some $c \in \mathbb{R}$. Now Proposition 5.1 yields the desired result.

Now suppose that $-m(2) = \infty$. Then by Lemma 7.3 we have that $|m(2 + i0)| = \infty$ too. Then by Lemma 7.5 we have

$$\frac{\|p_n(2)\|_L}{\|q_n(2)\|_L} \rightarrow 0$$

so we must have that $p_n(2)$ remains bounded. Thus, $p_n(2) = c\psi_s(n)$ for some $c \in \mathbb{R}$, so again we are done by Proposition 5.1. \square

8. PROOF OF THEOREM 1.1 ((1) \Rightarrow (2))

By Proposition 5.1 we see that all solutions at $E = \pm 2$ eventually satisfy $(\pm 1)^k \psi(k) > 0$. So by the Sturm oscillation theorem for Jacobi matrices (see chapter 4 of [19]), J has only finitely-many eigenvalues, all lying in $\mathbb{R} \setminus [-2, 2]$. So by Proposition 1.4 it suffices to prove the theorem when $\sigma(J) \subseteq [-2, 2]$, which we now assume.

Consider the values of the m function at $E = \pm 2$. We have four cases:

Case 1: $m(-2) = -m(2) = \infty$. As $Sz^{(e)}$ is onto, $d\nu \in \text{Ran}(Sz^{(e)})$, so choose

$$R_n(-2) = P_n(-2) \quad R_n(2) = P_n(2) \quad d\mu = [Sz^{(e)}]^{-1}(d\nu).$$

Case 2: $m(-2), -m(2) < \infty$. By Lemma 6.1, $d\nu \in \text{Ran}(Sz^{(o)})$, so choose

$$R_n(-2) = F_n(-2) \quad R_n(2) = F_n(2) \quad d\mu = [Sz^{(o)}]^{-1}(d\nu).$$

Case 3: $m(-2) = \infty, -m(2) < \infty$. By Lemma 6.1, $d\nu \in \text{Ran}(Sz^{(+)})$, so choose

$$R_n(-2) = P_n(-2) \quad R_n(2) = F_n(2) \quad d\mu = [Sz^{(+)}]^{-1}(d\nu).$$

Case 4: $m(-2) < \infty, -m(2) = \infty$. By Lemma 6.1, $d\nu \in \text{Ran}(Sz^{(-)})$, so choose

$$R_n(-2) = F_n(-2) \quad R_n(2) = P_n(2) \quad d\mu = [Sz^{(-)}]^{-1}(d\nu).$$

In any case, let α be the Verblunsky parameters associated to $d\mu$. By Proposition 7.1 we see that

$$\frac{R_{n+1}(-2)}{R_n(-2)} = 1 + l_1^2 \quad \frac{R_{n+1}(2)}{R_n(2)} = 1 + l_1^2.$$

Then by Proposition 6.5 we see that $\alpha \in l_1^2$. By Theorem 1.2 we see $\log w \in \dot{H}^{1/2}$, so by (1.6) we see $v \in \mathcal{W}$.

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